

Pre-sessional Mathematics for MSc Students

Class 1: Elements of Set Theory

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September 22, 2018

Set theory is an important branch of mathematics. It borders on mathematical logics and is vital for the very foundations of mathematical reasoning and human thought in general. A proper course in set theory should last for several terms and books on set theory may look intimidating.

Luckily we only need to scratch the surface of this monolith. In what may be called everyday life mathematicians are not concerned with arcane depths of set theory and mainly use it as a language.

There are plenty of books covering set theory to different depths. For example, [Ros06] (available from Royal Holloway library; any edition will do) covers the material from this class (and numerous other things less relevant to this MSc programme). Wikipedia has good articles on many set theory concepts.

1 Defining Sets

A *set* is a collection of objects. One may describe a finite set by enumerating all its elements, e.g.,

$$S_1 = \{1, 2, 3, 4, 5\} .$$

Sometimes infinite sets can also be described in this way by means of the magic "...", e.g.,

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

is the set of all positive integers (or *natural numbers*). When you use this notation, it should be clear what "... " conceals.

Note 1. Notation \mathbb{N} is often used for the set

$$\{0, 1, 2, 3, \dots\} .$$

Always check with your course lecturer whether 0 is a natural number! There is no universal consensus about this terminology issue. *Hint:* If he or she says, “Yes”, he or she is likely to be a logician by training.

Exercise 1. What is the standard name for the elements of the set $L = \{2, 4, 6, \dots\}$?

In many situations such simple tricks are not enough and definitions of the following kind are used:

$$S_2 = \{n \in \mathbb{N} \mid n^2 \leq 25\} .$$

This reads, “All positive integers n such that n^2 is less than or equal to 25”.

Such definitions are sometimes called *predicate definitions*: the condition $n^2 \leq 2$, which can be true or false, is a *predicate*.

2 Membership and Set Equality

Notation $x \in S$ reads, “ x belongs to S ” or “ x is an element of S ”. Sometimes the sign is inverted: $S \ni x$; this may be read as “ S contains x ”.

Two sets are *equal* if they have the same elements: $A = B$ if and only if for all x we have $x \in A$ if and only if $x \in B$. This can be shortened to

$$A = B \text{ iff for all } x \text{ we have } x \in A \text{ iff } x \in B$$

(“iff” is a popular shorthand for “if and only if”) or further shortened to

$$A = B \Leftrightarrow (\forall x(x \in A) \Leftrightarrow (x \in B)) .$$

Here \Leftrightarrow is an abbreviation for “if and only if” and \forall is an abbreviation for “for all”; the latter is a so called *quantifier*. Another quantifier is \exists and it reads “exists”. This is an awful way to write mathematical texts but mathematicians often do that.

For the sets S_1 and S_2 defined above we have $S_1 = S_2$. Indeed, there are only five positive integers less than or equal to 25. Those are 1, 2, 3, 4, and 5. To find out whether two sets are equal, you need to check if they contain the same elements. There is no unique procedure for this and this may take a deep mathematical investigation. For example, the set equality

$$\{n \in \mathbb{N} \mid \exists x, y, z \in \mathbb{N} : x^n + y^n = z^n\} = \{1, 2\} \tag{1}$$

(here the colon “:” reads “such that”) is the celebrated Fermat’s last theorem, which took centuries to prove.

Exercise 2. Wikipedia states Fermat’s last theorem as follows:

no three positive integers a , b , and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two.

Make sure you understand why set equality (1) means the same thing.

The usual direct way to show that two sets are not equal is to find an element that belongs to one set but not to the other. For example, take $S_3 = \{n \in \mathbb{N} \mid 0 < 3n + 1 < 20\}$. We have $S_1 \neq S_3$ because $6 \in S_3$ but $6 \notin S_1$ (the sign \notin reads “does not belong to” or “is not an element of”).

Exercise 3. Let $S_3 = \{n \in \mathbb{N} \mid 10n - 5 < 50\}$. Is $S_1 = S_3$? Justify your answer.

Exercise 4. Can you think of another way to show that two sets are not equal?

We say that $A \subseteq B$, or A is a *subset* of B if all elements of A belong to B . This may be written as

$$A \subseteq B \text{ iff } (\forall x(x \in A \Rightarrow (x \in B))) .$$

Here \Rightarrow reads “implies”.

Exercise 5. Is the following statement true or not? Justify your answer.

$$A = B \text{ iff } (A \subseteq B \text{ and } B \subseteq A)$$

Note 2. The concept of a set does not imply any *order* on its elements. $\{1, 2, 3, 4, 5\}$ is the same set as $\{5, 4, 3, 2, 1\}$ and it is usually wrong to speak of “the first element of the set” because there is no such thing. The concept of a set does not imply multiplicity either, so $\{2, 2\} = \{2\}$. If you need order or multiplicity (for example, having two identical pound coins in your pocket is not the same as having one coin), the set is just not a data structure representing your thoughts accurately. You need a different data structure, for example, a bag, or a sequence, or an array.

3 Some Common Sets

The following notation is often used in mathematical texts:

- \mathbb{N} the set of positive integers $\{1, 2, \dots\}$
(but note what was said about 0)
- \mathbb{Z} the set of all integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
(note the use of “ \dots ” at the front!)
- \mathbb{Q} the set of rational numbers $\{p/q \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
- \mathbb{R} the set of real numbers
(it is awfully hard to define a real number precisely)

4 Set Operations

The *union* of two (or more) sets contains all elements that belong to either set. The *intersection* of two (or more) sets contains all elements that belong to every set.

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6\}$. We have $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and $A \cap B = \{4, 5\}$.

By $|S|$ we mean the *cardinality* of a set S . If S is finite, $|S|$ is just the number of elements.

Exercise 6. Suppose that $|A| = 10$ and $|B| = 7$. What is $|A \cup B|$? Can you tell?

Exercise 7. Suppose that $|A| = 10$, $|B| = 7$, and $|A \cap B| = 3$. What is $|A \cup B|$? Can you tell?

The *set difference* $A \setminus B$ contains all elements that belong to A but not to B . Here the abbreviated statement may be handy:

$$A \setminus B = \{x \mid x \in A \text{ but } x \notin B\} .$$

Exercise 8. What is $(A \cap B) \cup (A \setminus B)$?

The *Cartesian product* $A \times B$ is the set of pairs where the first element belongs to A and the second belongs to B :

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\} .$$

If A and B are finite and

$$\begin{aligned} A &= \{a_1, a_2, \dots, a_n\} \\ B &= \{b_1, b_2, \dots, b_m\} , \end{aligned}$$

the Cartesian product can be visualised as

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & \dots & (a_1, b_m) \\ (a_2, b_1) & (a_2, b_2) & \dots & (a_2, b_m) \\ \vdots & \vdots & \ddots & \vdots \\ (a_n, b_1) & (a_n, b_2) & \dots & (a_n, b_m) \end{array}$$

Exercise 9. Let $|A| = n$ and $|B| = m$. What is $|A \times B|$? Can you tell?

In a Cartesian product order matters and $A \times B$ does not have to be the same set as $B \times A$. However it is the same if $A = B$. We can write $A^2 = A \times A$, $A^3 = A \times A \times A$ etc.

By definition \mathbb{R}^2 is the set of pairs of real numbers. Those can be understood as coordinates of points on a plane. So \mathbb{R}^2 is the two-dimensional plane. Similarly \mathbb{R}^3 is the set of triples of real numbers and can be identified with the three-dimensional space.

We can describe sets of points in the Euclidean space using the notation and tricks introduced above. For example,

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

the set of points at the distance of 1 from $(0, 0)$, is the circle (as a curve) and $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is the circle as a disk (mathematicians would call the former a sphere and the later a ball; the words apply to higher dimensions too.).

Exercise 10. Let $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$. Draw A , B , and $A \cap B$. Does $A \cap B = \{(x, y) \in \mathbb{R}^2 \mid (x - 0.5)^2 + (y - 0.5)^2 \leq 0.25\}$ hold? Justify your answer.

Exercise 11. Draw the set $\{(x, y) \in \mathbb{R}^2 \mid 2 \leq 2x + y \leq 6\}$.

Exercise 12. Let $C_z = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$. Describe this set.

Exercise 13. (Difficult.) Let $C_z = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$, $C_x = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 \leq 1\}$, and $C_y = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq 1\}$. Does $C_x \cap C_y \cap C_z = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ hold?

5 Functions

A function $f : A \rightarrow B$ (this reads “ f is a function from A to B ”) takes elements from A and maps them to B . The set A is called the *domain* of f and $f(a)$ should be defined on every $a \in A$. The set B is called the *co-domain* or sometimes *range* (the later is incorrect – see below).

Note 3. Mathematically a function is identified with its graph, i.e., the set $F = \{(a, b) \in A \times B \mid b = f(a)\}$. So a function is defined as a set $F \subseteq A \times B$ such that for every $a \in A$ there is b in B such that $(a, b) \in F$ and such b is unique. We will not need this advanced definition but digesting it is a very useful exercise and will enhance your understanding.

Here are some examples. Let the domain equal \mathbb{R} , the co-domain equal \mathbb{R} and f be given by $f(x) = x^2$. Technically whenever we speak of a function we must specify its domain and co-domain. In practice those are often dropped and we speak just of a function $f(x) = x^2$. This is OK as long as we do not need to talk about some properties of functions discussed below.

Here is another example. Let $g : S_1 \rightarrow \mathbb{N}$ (this notation is often used to specify the domain and the co-domain) be given by $g(n) = n^2$. Technically this is a different function because of the different domain and co-domain.

Sometimes functions are called mappings or transformations, especially when the co-domain is something more complex than \mathbb{R} (for example, $h(x, y) = (2x, 2y)$ is a transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^2$).

If $y = f(x)$, we say that y is the image of x (or “image of x under function/mapping/transformation f ”). If X is a subset of the domain, we can talk of the image of X under f : $f(X) = \{f(x) \mid x \in X\}$.

Exercise 14. What is the image of the circle $\{(x, y) \mid x^2 + y^2 = 1\}$ under the transformation $h(x, y) = (2x, 2y)$?

The image of the whole domain is called the *range* of f . For example, for $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ the range is the semi-infinite interval $[0, +\infty)$. The range is a subset of the co-domain, but does not have to equal it.

The following three cryptic terms are used to describe properties of functions. A function $f : A \rightarrow B$ is

- an *injection* iff for all $x_1, x_2 \in A$ if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$;
- a *surjection* iff for every $b \in B$ there is $a \in A$ such that $f(a) = b$;
- a *bijection* iff it is an injection *and* a surjection.

Exercise 15. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$. Is it an injection? Is it a surjection? Is it a bijection?

The definition of the surjection essentially says that the range should equal the co-domain; this is a reason to distinguish these two concepts.

A bijection is also called one-to-one correspondence.

Exercise 16. Let $|A| = n$ and $f : A \rightarrow B$ is a bijection. What is $|B|$?

Exercise 17. Let $|A| = n$ and $f : A \rightarrow B$ is an injection. Is it possible that

- $|B| < n$,
- $|B| = n$, or
- $|B| > n$?

Exercise 18. Let $|A| = n$ and $f : A \rightarrow B$ is a surjection. Is it possible that

- $|B| < n$,

- $|B| = n$, or
- $|B| > n$?

Let $f : A \rightarrow B$ be a function. A function $f^{-1} : B \rightarrow A$ is called an *inverse function* to f if for all $x \in A$ we have $f^{-1}(f(x)) = x$ and for all $y \in B$ we have $f(f^{-1}(y)) = y$. (The notation f^{-1} can be confused with $1/f$ so care should be taken.)

It is easy to see that an inverse function exists if and only if f is a bijection. Indeed, if f is not an injection and $f(x_1) = f(x_2) = y$ for some $x_1 \neq x_2$, then f^{-1} would not know where to send y to. If, say, $f^{-1}(y) = x_2$, then $f^{-1}(f(x_1)) = f^{-1}(y) = x_2 \neq x_1$. If f is not a surjection, there is $y \in B$ such that there is no $x \in A$ such that $f(x) = y$. Again, f^{-1} would not know where to send y to. If, say, $f^{-1}(y) = x_0$, then $f(x_0) \neq y$ and $f(f^{-1}(y)) = f(x_0) \neq y$. However, if f is a bijection, then for every $y \in B$ there is $x \in A$ such that $f(x) = y$ and this x is unique. So we can let $f^{-1}(y) = x$ and thus define an inverse function. Besides, there is only one possible choice for $f^{-1}(y)$ and therefore the inverse function is unique.

If $f : A \rightarrow B$ is an injection, then f considered as a function $A \rightarrow f(A)$ is a bijection and has an inverse.

Exercise 19. Is there an inverse function to $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$? If so, find it. If not, explain why.

Exercise 20. Is there an inverse function to $f : \mathbb{R} \rightarrow [0, +\infty)$ given by $f(x) = x^2$? If so, find it. If not, explain why.

Exercise 21. Is there an inverse function to $f : [0, +\infty) \rightarrow [0, +\infty)$ given by $f(x) = x^2$? If so, find it. If not, explain why.

References

- [Ros06] K. H. Rosen. *Discrete mathematics and its applications*. McGraw-Hill, 6th edition, 2006.