On games of continuous and discrete randomized forecasting

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Game-theoretic approach

Game-theoretic approach to forecasting and probability

BINARY FORECASTING GAME II

FOR $n = 1, 2, \ldots$

**Skeptic** announces $S_n : [0, 1] \rightarrow \mathbb{R}$ (set of all real numbers).

**Forecaster** announces a probability distribution $P_n \in \mathcal{P}[0, 1]$.

**Reality** announces $\omega_n \in \{0, 1\}$.

**Forecaster** announces $f_n : [0, 1] \rightarrow \mathbb{R}$ such that
\[
\int f_n(p) P_n(dp) \leq 0.
\]

**Random number generator** announces $p_n \in [0, 1]$.

Sceptic updates his total gain $\mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n)$.

Forecaster updates his total gain $\mathcal{F}_n = \mathcal{F}_{n-1} + f_n(p_n)$.

ENDFOR
Restriction on Skeptic: Skeptic must choose the $S_n$ so that his total gain $\mathcal{K}_n$ is nonnegative for all $n$ no matter how the other players move; $\mathcal{K}_0 = 1$.

Restriction on Forecaster: Forecaster must choose the $P_n$ and $f_n$ so that his total gain $\mathcal{F}_n$ is nonnegative for all $n$ no matter how the other players move; $\mathcal{F}_0 = 1$.

Winners:

Forecaster wins if either (i) his total gain $\mathcal{F}_n$ is unbounded or (ii) Skeptic’s total gain $\mathcal{K}_n$ stays bounded; otherwise the other players win.
Binary Forecasting Game II: Who win?

Theorem

Forecaster has a winning strategy in Binary Forecasting Game II.


The von Neumann minimax theorem is used on each round $n$. 
Sketch of the proof

Zero-sum auxiliary game: **Forecaster** announces $p_n \in [0, 1]$, **Nature** announces $\omega_n \in \{0, 1\}$.

$$F(\omega_n, p_n) = S(p_n)(\omega_n - p_n) - \text{Forecaster's loss (Nature gain)}$$

For each **Nature's** mixed strategy $Q_n \in \mathcal{P}\{0, 1\}$ a **Forecaster's** pure strategy $p_n = Q\{1\}$ exists such that

$$F(Q_n, P_n) = E_{Q,P}(F(\omega_n, p_n)) = \int S_n(p_n)))(\omega - p_n)dQ = 0.$$

Hence, $\max_Q \min_P F(Q, P) \leq 0$. 
After discretization by $P$

$$\max_Q \min_P F(Q, P) \leq \Delta.$$ 

By minimax theorem

$$\min_P \max_Q F(Q, P) = \max_Q \min_P F(Q, P) \leq \Delta.$$ 

Equivalently, $P_\Delta$ exists such that 

$$\forall Q : F(Q, P_\Delta) \leq \Delta,$$ or

**Forecaster** has a mixed strategy $P_\Delta$ on a discrete set such that

$$\int S_n(p)(\omega_n - p)P_\Delta(dp) \leq \Delta$$

for $\omega_n = 0$ and $\omega_n = 1.$
For $\Delta \to 0$ we obtain

$$\int S_n(p)(\omega_n - p)P_n(dp) \leq 0$$

for $\omega_n = 0$ and $\omega_n = 1$,

$P_n$ is a limit point of $\{P_{\Delta}\}$ in the weak topology.
Forecaster’s winning strategy:

Forecaster’s Move 1: $P_n$

Forecaster’s Move 2: $f_n(p) = S_n(p)(\omega_n - p)$

Then $\mathcal{F}_n = \mathcal{K}_n$.

Forecaster wins since $\sup_n \mathcal{K}_n < \infty$ or $\sup_n \mathcal{F}_n = \infty$
Universal forecasting requires unrestrictedly increasing degree of accuracy.

We present some results showing that discrete universal forecasting is impossible.
Level of discreteness

Measure $P_n$ is concentrated on a finite subset $D_n \subset [0, 1]$ 
$D_n = \{p_{n,1}, \ldots, p_{n,m_n}\}$.

$\Delta_n = \inf\{|p_{n,i} - p_{n,j}| : i \neq j\}$;

$\Delta = \liminf_{n \to \infty} \Delta_n$ is called the strategy’s level of discreteness.

A typical example is the uniform rounding of $[0, 1]$. 
PROBABILISTIC BINARY FORECASTING GAME II

FOR $n = 1, 2, \ldots$

**Skeptic** announces $S_n : [0, 1] \to \mathbb{R}$.

**Forecaster** announces a probability distribution $P_n \in \mathcal{P}[0, 1]$.

**Reality** announces $\omega_n \in \{0, 1\}$.

**Random Number Generator** announces $p_n \in [0, 1]$.

**Skeptic** updates his total gain

$$K_n = K_{n-1} + S_n(p_n)(\omega_n - p_n).$$

ENDFOR

Restriction on Skeptic: Skeptic must choose the $S_n$ so that his total gain $K_n$ is nonnegative for all $n$ no matter how the other players move; $K_0 = 1$.

Realty and Skeptic win if Skeptic’s total gain $K_n$ is unbounded; otherwise Forecaster wins.
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Probability-game-theoretic form

$Pr$ – overall probability distribution on infinite paths $p_1, p_2, \ldots$ of Forecaster’s moves (there exists by Ionescu-Tulcea theorem)

**Theorem**

*If Forecaster uses a randomized strategy with a positive level of discreteness then Realty and Skeptic win in Probabilistic Binary Forecasting Game II with $Pr$-probability 1. Otherwise, Forecaster wins with $Pr$-probability 1.*
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Game-theoretic symmetric forms

Game-theoretic counterparts
Symmetric Binary Forecasting Game II: Protocol

SYMOMETRIC BINARY FORECASTING GAME II

FOR $n = 1, 2, \ldots$

**Skeptic** announces $S_n : [0, 1] \rightarrow \mathbb{R}$ (set of all real numbers).

**Forecaster** announces a probability distribution $P_n \in \mathcal{P}[0, 1]$.

**Reality** announces $\omega_n \in \{0, 1\}$.

**Forecaster** announces $f_n : [0, 1] \rightarrow \mathbb{R}$ such that
\[
\int f_n(p)P_n(dp) \leq 0.
\]

**Skeptic** announces $h_n : [0, 1] \rightarrow \mathbb{R}$ such that
\[
\int h_n(p)P_n(dp) \leq 0.
\]

**Random Number Generator** announces $p_n \in [0, 1]$.

**Skeptic** updates both his total gains:
\[
K_n = K_{n-1} + S_n(p_n)(\omega_n - p_n).
\]
\[
G_n = G_{n-1} + h_n(p_n) \text{ (statistical gain)}.
\]

**Forecaster** updates his total statistical gain:
\[
F_n = F_{n-1} + f_n(p_n).
\]

ENDFOR
Restriction 1 on Skeptic: Skeptic must choose the $S_n$ so that his total gain $\mathcal{K}_n$ is nonnegative for all $n$ no matter how the other players move; $\mathcal{K}_0 = 1$.

Restriction 2 on Skeptic: Skeptic must choose the $h_n$ and $S_n$ so that his total gain $\mathcal{G}_n$ is nonnegative for all $n$ no matter how the other players move; $\mathcal{G}_0 = 1$.

Restriction on Forecaster: Forecaster must choose the $P_n$ and $f_n$ so that his total gain $\mathcal{F}_n$ is nonnegative for all $n$ no matter how the other players move; $\mathcal{F}_0 = 1$.
Symmetric Binary Forecasting Game II: Protocol

Three parties:

1) **Sceptic** and **Realty** against 2) **Forecaster**

3) **Random Number Generator** – neutral player
Random Number Generator is **fair** in the game if both statistical total gains are bounded $\sup_n G_n < \infty$ and $\sup_n F_n < \infty$.

Assume that Random Number Generator is **fair**. Winners in this case:

**Sceptic** and **Realty** win if the Skeptic’s total gain is unbounded: $\sup_n K_n = \infty$; otherwise **Forecaster** wins.
The following theorem shows that in case where Random Number Generator is fair Forecaster wins if and only if it can use a randomized strategy with zero level of discreteness.

**Theorem**

Assume Random Number Generator is fair. If Forecaster’s uses a randomized strategy with a positive level of discreteness.\(^a\) then Realty and Skeptic win in the Symmetric Binary Forecasting Game II. Otherwise, Forecaster wins.

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\(^a\)A value of this level of discreteness is unknown for Realty and Skeptic.
Symmetric Binary Forecasting Game II: Protocol

Two parties

1) **Sceptic, Realty, and Random Number Generator**

against

2) **Forecaster**
ASYMMETRIC BINARY FORECASTING GAME II – Simplification

FOR $n = 1, 2, \ldots$

**Skeptic** announces $S_n : [0, 1] \rightarrow \mathbb{R}$ (set of all real numbers).

**Forecaster** announces a probability distribution $P_n \in \mathcal{P}[0, 1]$.

**Reality** announces $\omega_n \in \{0, 1\}$.

**Forecaster** announces $f_n : [0, 1] \rightarrow \mathbb{R}$ such that

\[ \int f_n(p)P_n(dp) \leq 0. \]

**Skeptic** announces $h_n : [0, 1] \rightarrow \mathbb{R}$ such that \[ \int h_n(p)P_n(dp) \leq 0. \]

**Random Number Generator** announces $p_n \in [0, 1]$.

**Skeptic** updates both his gains

\[ \mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n) + h_n(p_n). \]

**Forecaster** updates his total statistical gain:

\[ \mathcal{F}_n = \mathcal{F}_{n-1} + f_n(p_n). \]

ENDFOR
ASYMMEtRIC BINARY FORECASTING GAME II

\( \mathcal{K}_0 = 1. \)

FOR \( n = 1, 2, \ldots \)

Skeptic announces \( S_n : [0, 1] \rightarrow \mathbb{R}. \)

Forecaster announces a probability distribution \( P_n \in \mathcal{P}[0, 1]. \)

Reality announces \( \omega_n \in \{0, 1\}. \)

Skeptic announces \( h_n : [0, 1] \rightarrow \mathbb{R} \) such that \( \int h_n(p)P_n(dp) \leq 0. \)

Random Number Generator announces \( p_n \in [0, 1]. \)

Skeptic updates his total gain

\( \mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n) + h_n(p_n). \)

ENDFOR

Realty and Skeptic win if Skeptic’s total gain \( \mathcal{K}_n \) is unbounded; otherwise Forecaster and Random Number Generator win.
Main result

**Theorem**

Assume Forecaster’s uses a randomized strategy with a positive level of discreteness. Then Realty and Skeptic win in the Asymmetric Binary Forecasting Game II.
Sketch of the proof

**Strategy for Realty:** at any step $n$ Realty announces an outcome

$$\omega_n = \begin{cases} 
0 & \text{if } P_n((0.5, 1]) > 0.5 \\
1 & \text{otherwise.}
\end{cases}$$

**Strategy for Sceptic:** Move 1 and Move 2 (below).
Skeptic’s capitals:

Sceptic’s capital for Move 1:
\[ \mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n) \]

Sceptic’s (statistical) capital for Move 2:
\[ \mathcal{G}_n = \mathcal{G}_{n-1} + g_n(p_n) \text{ for all } n > 0. \]

Forecaster’s (statistical) capital for Move 2:
\[ \mathcal{F}_n = \mathcal{F}_{n-1} + f_n(p_n) \text{ for all } n > 0. \]

Initially, \( \mathcal{K}_0 = 1, \mathcal{G}_0 = 1, \) and \( \mathcal{F}_0 = 1. \)
\[
\vartheta_{n,1} = \sum_{j=1}^{n} \xi(p_j > 0.5)(\omega_j - p_j)
\]

\[
\vartheta_{n,2} = \sum_{j=1}^{n} \xi(p_j \leq 0.5)(\omega_j - p_j)
\]

where \(\xi(\text{true}) = 1\) and \(\xi(\text{false}) = 0\).

We have \(\vartheta_{n,2} - \vartheta_{n,1} = \sum_{j=1}^{n} g_j(p_j)\), where

\[
g_j(p) = \xi(p \leq 0.5)(\omega_j - p) - \xi(p > 0.5)(\omega_j - p).
\]

For any discrete Forecaster’s strategy \(\{P_j\}\), in the mean:

\[
E(\vartheta_{n,2} - \vartheta_{n,1}) = \sum_{j=1}^{n} E_{P_j}(g_j) \geq 0.5\Delta n.
\]
Since Random Number Generator is fair, $\sup \mathcal{G}_n < \infty$.

**Move 2 of Sceptic’s strategy forces:**

$$\sup_n \mathcal{G}_n < \infty \Rightarrow \liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) \geq -\varepsilon.$$  

Then

$$\liminf_{n \to \infty} \frac{1}{n} (\vartheta_{n,2} - \vartheta_{n,1}) \geq 0.5\Delta - \varepsilon,$$

where $\varepsilon > 0$ is arbitrary small.
Move 1 of Sceptic’s strategy forces:

\[ 2 \frac{\ln Q_n}{n} \geq \varepsilon (\vartheta_{n,2} - \vartheta_{n,1}) - 2\varepsilon^2 \geq \varepsilon (0.5\Delta - \varepsilon) - 2\varepsilon^2 > \varepsilon^2 \]

for infinitely many \( n \), where \( \varepsilon > 0 \) is arbitrary small fixed real number (tuned in the game to be much smaller than \( \Delta : \varepsilon < \Delta/8 \)).

Therefore,

\[ \limsup_{n \to \infty} \frac{\ln Q_n}{n} > \varepsilon / 2. \]

Hence, Sceptic’s capital is unbounded

\[ \sup_n \mathcal{K}_n = \infty. \]
**Calibration**: Kakade and Foster’ result - 2004

**Theorem**

For any sequence of outcomes $\omega_1 \omega_2 \ldots$, an observer can only randomly round the deterministic forecast up to $\Delta$ in order to calibrate with the internal probability $1$:

\[
\left| \frac{1}{n} \sum_{i=1}^{n} I(p_i)(\omega_i - p_i) \right| \leq \Delta
\]

for all $n$, where $\Delta$ is the calibration error, $I(p)$ is the indicator function of an arbitrary subinterval of $[0, 1]$. 
A lower bound of calibration error:

**Corollary**

Assume Forecaster uses a randomized strategy with a positive level of discreteness \( \Delta \). Then Realty (without using information on a value of \( \Delta \)) can announce an infinite binary sequence \( \omega_1 \omega_2 \ldots \) such that one of two possibilities holds:

\[
\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{j=1}^{n} I(p_j > 0.5)(\omega_j - p_j) \right| \geq 0.25\Delta
\]

\[
\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{j=1}^{n} I(p_j \leq 0.5)(\omega_j - p_j) \right| \geq 0.25\Delta
\]
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More details:
Auxiliary Skeptic’s strategies for Move 1:

\[ S_{n}^{1,k}(p) = -\varepsilon_{k} \mathcal{Q}_{n-1}^{1,k} \xi(p > 0.5), \quad (1) \]
\[ S_{n}^{2,k}(p) = \varepsilon_{k} \mathcal{Q}_{n-1}^{2,k} \xi(p \leq 0.5), \quad (2) \]

where \( \xi(true) = 1, \xi(false) = 0, \) and \( n \geq 1 \)

Auxiliary Skeptic’s capital for Move 1:

\[ \mathcal{Q}_{n}^{1,k} = \mathcal{Q}_{n-1}^{1,k} + S_{n}^{1,k}(p_n)(\omega_n - p_n)), \]
\[ \mathcal{Q}_{n}^{2,k} = \mathcal{Q}_{n-1}^{2,k} + S_{n}^{2,k}(p_n)(\omega_n - p_n)). \]
Skeptic’s strategy for Move 1:

$$S_n(p) = \frac{1}{2} (S_n^1(p) + S_n^2(p)),$$

where

$$S_n^1(p) = \sum_{k=1}^{\infty} \epsilon_k S_n^{1,k}(p)$$
$$S_n^2(p) = \sum_{k=1}^{\infty} \epsilon_k S_n^{2,k}(p).$$

Skeptic’s capital for Move 1:

$$Q_n = \frac{1}{2} \sum_{k=1}^{\infty} \epsilon_k (Q_n^{1,k} + Q_n^{2,k}).$$
Define $g_n(p) = \xi(p \leq 0.5)(\omega_n - p) - \xi(p > 0.5)(\omega_n - p)$

**Auxiliary Skeptic’s strategy and capital for Move 2:**

Define recursively by $n$, $\mathcal{F}_0^k = 1$, $g_0^k(p) = 0$;

$$g_n^k(p) = -\varepsilon_k \mathcal{F}_{n-1}^k (g_n(p) - E_{P_n}(g_n)),$$

$$\mathcal{F}_n^k = \mathcal{F}_{n-1}^k + g_n^k(p_n)$$

for $n \geq 1$, where $\varepsilon_k = 2^{-k}$ and $P_n$ – Forecaster’s move.
Skeptic’s strategy for Move 2:

\[ h_n(p) = \sum_{k=1}^{\infty} \varepsilon_k g_n^k(p). \]

By definition \( \int h_n(p) P_n(dp) \leq 0. \)

Skeptic’s (statistical) capital for Move 2:

\[ G_n = \sum_{k=1}^{\infty} \varepsilon_k G_n^k. \]

Also, \( G_n \geq 0 \) for all \( n \).
We have for each $k$,

$$\ln \mathcal{G}_n^k \geq -\varepsilon_k \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) - n\varepsilon_k^2.$$ 

Since $\sup_n \mathcal{G}_n < C$, we have for any $k$

$$\frac{1}{n} \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) \geq \frac{-\ln C + \ln(\varepsilon_k)}{n\varepsilon_k} - \varepsilon_k \geq -2\varepsilon_k$$

Hence,

$$\frac{1}{n} \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) \geq -2\varepsilon_k$$

for all sufficiently large $n$. 
Result of Sceptic’s Move 2

\[
\frac{1}{n} (\vartheta_{n,2} - \vartheta_{n,1}) = \frac{1}{n} \sum_{j=1}^{n} g_j(p_j) \geq \frac{1}{n} \sum_{j=1}^{n} E_{P_j}(g_j) - 2\varepsilon_k \geq 0.5\Delta - 2\varepsilon_k.
\]
Sceptic’s Move 1

\[
\ln Q_{1,n}^k \geq -\varepsilon_k \vartheta_{n,1} - \varepsilon_k^2 n, \\
\ln Q_{2,n}^k \geq \varepsilon_k \vartheta_{n,2} - \varepsilon_k^2 n.
\]

Hence,

\[
\frac{\ln Q_{1,n}^k + \ln Q_{2,n}^k}{n} \geq \varepsilon_k \frac{1}{n} (\vartheta_{n,2} - \vartheta_{n,1}) - 2\varepsilon_k^2 \geq \varepsilon_k (0.5\Delta - 2\varepsilon_k) - 2\varepsilon_k^2 = 0.5\varepsilon_k \Delta - 2\varepsilon_k^2 \geq 2\varepsilon_k^2
\]

for all sufficiently large \(n\), where \(\varepsilon_k \leq \frac{1}{8} \Delta\).
From this, we obtain

$$\limsup_{n \to \infty} \frac{\ln q_{i,k}^n}{n} \geq \epsilon_k^2$$

for $i = 1$ or $i = 2$.

Hence,

$$\sup_n q_n = \infty$$

no matter how Forecaster moves if Realty uses her strategy defined above.