Zero-sum two person game, Kullback-Leibler information, sequential hypothesis testing, in relation to Shafer-Vovk theory

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1.1 Formulation

The unit zero-sum two-person game:

- Pay-off matrix $A = \{a_{ij}\} : k \times m$.
- The first player: Skeptic
- The second player: Reality
- There are $k$ items Skeptic can bet and $m$ outcomes Reality can choose.
- The value of the game $v^* = v^*(A)$. 
Protocol of the betting game

Protocol:
\( \mathcal{K}_0 = 1 \)
FOR \( n = 1, 2, \ldots \)
  
  Skeptic announces \( M_{ni} \geq 0, \ i = 1, \ldots, k \).
  
  Reality announces \( j_n \in \{1, \ldots, m\} \).

  \[ \mathcal{K}_n = \mathcal{K}_{n-1} + \sum_{i=1}^{k} M_{ni} x_{ni}, \text{ where } x_{ni} = a_{ijn}. \]

END FOR

(Since \( M_{ni} \geq 0 \), we are considering a one-sided game.)
- The set of optimal strategies for Skeptic in the unit game:

\[ Q^* = \{ \alpha \}, \ \alpha = (\alpha_1^*, \ldots, \alpha_k^*)', \ \alpha_i^* \geq 0, \sum_{i=1}^{k} \alpha_i^* = 1. \]

- The set of optimal strategies for Reality:

\[ P^* = \{ p \}, \ p = (p_1^*, \ldots, p_m^*)', \ p_j^* \geq 0, \sum_{j=1}^{m} p_j^* = 1. \]
The value $v^*$ of the game:

For $\alpha^* \in Q^*$, \( \sum_{i=1}^{k} a_{ij} \alpha_i^* \geq v^* \), \( \forall j \in \{1, \ldots, m\} \) \hspace{1cm} (1)

For $p^* \in P^*$, \( \sum_{j=1}^{m} a_{ij} p_j^* \leq v^* \), \( \forall i \in \{1, \ldots, k\} \) \hspace{1cm} (2)

- If $v^* > 0$, then Skeptic can make $K_n \to \infty$ in one step, by (1).

- If $v^* < 0$, then Reality can make $K_n \leq 0$ as long as Skeptic keeps betting positive amount, which is bounded away from 0.

- We call the betting game **fair** if $v^* = 0$. 


1.2 Some definitions on the unit game

From now on we assume $v^* = 0$.

**Definition 1** The unit game is regular if for every $j$, there exists $p^* \in P^*$ such that $p_j^* > 0$.

**Proposition 1** If the unit game is non-regular (i.e. if for some $j_0$, $p_{j_0}^* = 0$, $\forall p^* \in P^*$), then there exists $\alpha^* \in Q^*$ such that

\[
\sum_{i=1}^{k} a_{ij_0} \alpha_i^* > 0, \quad \sum_{i=1}^{k} a_{ij} \alpha_i^* \geq 0, \forall j.
\]
Proposition 2  If the unit game is regular, then there exists $p^* \in P^*$ such that $p^*_j > 0, \forall j$.

Definition 2  The unit game is non-redundant, if it is regular and $p^* \in P^*$ is unique.

(corresponds to “complete market” in mathematical finance)

Proposition 3  If the unit game is non-redundant with $\{p^*\} = P^*$, then $\sum_{i=1}^{k} a_{ij} \alpha^*_i = 0, \forall j, \forall \alpha^* \in Q^*$. 
Definition 3  The $j_0$-th outcome is weakly dominated if there is $p$ such that $p_j \geq 0$, $\forall j$, $\sum_j p_j = 1$, $p_{j_0} = 0$, and

$$
\sum_{j=1}^{m} a_{ij} p_j \leq a_{ij_0}, \ \forall i.
$$

(3)

The $j_0$-th outcome is strongly dominated if the inequality in (3) is strict for all $i$.

Proposition 4  If the unit game is regular, no outcome for Reality is strongly dominated, and when it is non-redundant, no outcome for Reality is weakly dominated.
Proposition 5  If the $j_0$-th outcome for Reality is weakly dominated, the value of the unit game does not change when the $j_0$-th outcome is deleted from the game.

Definition 4  The $i_0$-th item Skeptic is weakly dominated if there is $\{\alpha_i\}$ such that $\alpha_i \geq 0$, $\forall i$, $\sum_i \alpha_i = 1$, $\alpha_{i_0} = 0$ and

$$\sum_{i=1}^{k} a_{ij} \alpha_i \geq a_{i_0 j}, \ \forall j.$$

It is strongly dominated if the inequalities are strict for all $j$. 
1.3 Price of an additional item

- Let $A$ be a pay-off matrix with $v^*(A) = 0$.
- Let $b = \{b_j\}$ be an additional item for Skeptic. Assume $b_j > 0, \exists j$.
- For $\pi > 0$ define an augmented payoff matrix

$$\tilde{A} = \begin{pmatrix}
\frac{b_1}{\pi} - 1 & \cdots & \frac{b_m}{\pi} - 1 \\
A & & \\
\end{pmatrix}$$
Definition 5 \( \pi \) is a proper price of \( b \), if \( v^*(\tilde{A}) = 0 \) and there is an optimal strategy for Skeptic in \( \tilde{A} \) for which \( a_{k+1} > 0 \).

Proposition 6 \( v^*(\tilde{A}) = 0 \) if and only if

\[
\pi \geq \pi^* = \inf_{p^* \in P^*} \sum_{j} b_j p_j^*.
\]

(inf and sup are attained because our setup is finite-dimensional.)

Proposition 7 Let \( \pi \) be the proper price.

\[
\pi \leq \bar{\pi}^* = \sup_{p^* \in P^*} \sum_{j} b_j p_j^*.
\]
Proposition 8  $\bar{\pi}^*$ can be expressed as the upper expectation

$$\bar{\pi}^* = \min \{ \pi \mid \exists \{\alpha_i \geq 0\} \text{ s.t. } \sum_{i=1}^{k} a_{ij} \alpha_i \geq b_j - \pi, \forall j \}.$$
1.4 Expanded form of multi-stage betting game

Consider the $N$-step betting game in the multiplicative form $\alpha_{ni} = M_{ni}/\mathcal{K}_{n-1}$:

$\mathcal{K}_0 = 1$.

FOR $n = 1, \ldots, N$.

- Skeptic announces $\alpha_{ni} \geq 0$, $i = 1, \ldots, k$.
- Reality announces $j_n \in \{1, \ldots, m\}$.

$\mathcal{K}_n = \mathcal{K}_{n-1}(1 + \sum_{i=1}^{k} \alpha_{ni}x_{ni})$.

END FOR
• Skeptic chooses

\[ I = \{i_1, i_2(j_1), i_3(j_1, j_2), \ldots, i_N(j_1, \ldots, j_{N-1})\} \]

• Reality chooses

\[ J = \{j_1, j_2, \ldots, j_N\} \]

• Expanded payoff matrix \( \{\tilde{A}_{IJ}\} \):

\[ \tilde{A}_{IJ} = (1 + a_{i_1j_1})(1 + a_{i_2(j_1)j_2}) \ldots (1 + a_{i_N(j_1, \ldots, j_{N-1})j_N}) - 1 \]

• Optimal strategies: \( \{\alpha_I^*\} \in \tilde{Q}^*, \{p_J^*\} \in \tilde{P}^* \).

• If \( v^*(A) = 0 \) then \( v^*(\{\tilde{A}_{IJ}\}) = 0 \).
Theorem 1  Strategies for two players are optimal if and only if all the conditional strategies (conditional probabilities) are optimal for the unit game, i.e.

$$\{\tilde{\alpha}_I^*\} \in \tilde{Q}^* \iff \{\alpha_{i_k(j_1,\ldots,j_{k-1})}\} \in Q^*, \quad \forall (j_1, \ldots, j_{k-1}), \forall k.$$ 

$$\{\tilde{p}_J^*\} \in \tilde{P}^* \iff \{p_{j_k(j_1,\ldots,j_{k-1})}\} \in P^*, \quad \forall (j_1, \ldots, j_{k-1}), \forall k.$$
Additional item ("derivative") in $N$-step betting game. $\phi(J) = \phi(j_1, \ldots, j_N)$

- Let

$$\bar{\pi}_\phi^* = \sup_{\{p_j^*\} \in \hat{P}^*} E\{p_j^*\}(\phi(J)), \quad \underline{\pi}_\phi^* = \inf_{\{p_j^*\} \in \hat{P}^*} E\{p_j^*\}(\phi(J))$$

- These are calculated by backward induction.

**Definition 6**

$$\bar{\pi}_\phi^*(j_1, \ldots, j_k) = \sup_{p_{k+1}, \ldots, p_N^* \in P^*} E(\phi(j_1, \ldots, j_k, j_{k+1}, \ldots, j_N))$$

$$\underline{\pi}_\phi^*(j_1, \ldots, j_k) = \inf_{p_{k+1}, \ldots, p_N^* \in P^*} E(\phi(j_1, \ldots, j_k, j_{k+1}, \ldots, j_N))$$
• Then

\[ \bar{\pi}_\phi^*(j_1, \ldots, j_k) = \sup_{p_{k+1}^* \in P^*} E(\bar{\pi}_\phi^*(j_1, \ldots, j_k, j_{k+1})) \]

\[ \underline{\pi}_\phi^*(j_1, \ldots, j_k) = \inf_{p_{k+1}^* \in P^*} E(\bar{\pi}_\phi^*(j_1, \ldots, j_k, j_{k+1})) \]

• The case of non-redundant unit game.

If the unit game \( A \) is non-redundant with the unique optimal strategy \( p^* \), then the unique optimal strategy \( \tilde{p}^* \) for the expanded game is the \( N \)-fold direct product (namely, i.i.d.) 

\( \tilde{p}^* = p^* \times \cdots \times p^* \) of \( p^* \) and

\[ \bar{\pi}_\phi^* = \underline{\pi}_\phi^* = E_{\tilde{p}^*}(\phi(J)) = \pi^*_\phi. \]
Sequence of $N$-step games ($N \to \infty$)

- Consider a sequence of $N$-step games with the unit game $A_N = A/\sqrt{N} = \{a_{ij}/\sqrt{N}\}$
- Let

$$\phi(j_1, \ldots, j_N) = \psi\left(\sum_{i=1}^{k} c_i \sum_{n=1}^{N} a_{ijn}/\sqrt{N}\right),$$

where $\psi \in C^3(\mathbb{R}^1)$. 
• Let $A$ be non-redundant. Then as $N \to \infty$

$$\pi^*_\phi \to \int_{-\infty}^{\infty} \psi(u) \frac{1}{\sqrt{2\pi V}} \exp\left(-\frac{u^2}{2V}\right) du,$$

where $V = \sum_{i,i'} c_i c_{i'} \sum_j a_{ij} a_{i'j} p^*_j$.

• Suppose that $A$ is not non-redundant ("incomplete case"). Let

$$\bar{V} = \sup_{p^* \in P^*} \sum_{i,i'} c_i c_{i'} \sum_j a_{ij} a_{i'j} p^*_j$$

$$V = \inf_{p^* \in P^*} \sum_{i,i'} c_i c_{i'} \sum_j a_{ij} a_{i'j} p^*_j$$
Then

\[
\tilde{\pi}_\phi^* \rightarrow \int_{-\infty}^{\infty} \psi(u) \frac{1}{\sqrt{2\pi \tilde{V}}} \exp\left(-\frac{u^2}{2\tilde{V}}\right) du, \quad \text{if } \psi \text{ is convex}
\]

\[
\bar{\pi}_\phi^* \rightarrow \int_{-\infty}^{\infty} \psi(u) \frac{1}{\sqrt{2\pi V}} \exp\left(-\frac{u^2}{2V}\right) du, \quad \text{if } \psi \text{ is concave}
\]
If $\psi$ is \textit{neither convex nor concave}, we have no explicit expression for the limiting formula for the upper and lower prices.

- Define

$$\bar{\pi}(x, \frac{n}{N}) = \sup_{p^*_n, \ldots, p^*_N} E(\psi(x + \sum_i c_i \sum_{l=n+1}^N a_{ijl}/\sqrt{N})).$$

- Under some regularity conditions

$$\bar{\pi}(x, n/N) \to \pi^*(x, t), \quad 0 \leq t \leq 1.$$
\[ \frac{\partial \pi^*}{\partial t} = \frac{1}{2} \sigma_x^2 \frac{\partial^2 \pi^*}{\partial x^2}, \text{ where } \sigma_x^2 = \begin{cases} \bar{V} & \text{if } \frac{\partial^2 \pi^*}{\partial x^2} \geq 0 \\ V & \text{if } \frac{\partial^2 \pi^*}{\partial x^2} < 0 \end{cases} \]

- \( \pi^*(x, t) \) is the solution of the following partial differential equation

- This partial differential equation is sometimes called “Black-Scholes-Barenblatt equation”. When \( V = 0 \), the solution for the equation exists in the sense of viscosity solution.

[This is discussed in a manuscript in preparation.]
2. Kelly’s strategy and Kullback-Leibler information
Kelly’s strategy and KL-information

- If Reality adopts a non-optimal strategy, then Skeptic can exploit the error.
- Assume that Reality’s strategy $p_0 = \{p_j^0\}$ is not in $P^*$. (common for each round).
- Skeptic chooses Kelly’s strategy which maximizes

$$E_{p_0}(\log K_1) = \sum_{j=1}^{m} p_j^0 \log(1 + \sum_{i=1}^{k} \alpha_i p_j^0).$$
• Assume that the game is regular and \( p_j^0 > 0 \) for all \( j \). Then Skeptic’s best strategy \( \alpha_i^* \) satisfies

\[
\sum_{j=1}^{m} \frac{a_{ij}}{1 + \sum_{h=1}^{k} \alpha_h^* a_{hj}} p_j^0 \leq 0, \ \forall i, \ \text{and}
\]

\[
= 0 \ \text{if} \ \alpha_i^* > 0
\]

• By defining

\[
p_j^* = \frac{1}{1 + \sum_{h=1}^{k} \alpha_h^* a_{hj}} p_j^0
\]

we have

\[
\sum_{j=1}^{m} a_{ij} p_j^* \leq 0, \ \forall i, \ \sum_{j=1}^{m} p_j^* = 1.
\]
• Therefore $\{p_j^*\} \in P^*$.

**Theorem 2**  \[
\log \mathcal{K}_1 = 1 + \sum_h \alpha_h^* a_{hj} = \log (p_j^0 / p_j^*) \quad \text{and} \\
E_{p_0}(\log \mathcal{K}_1) = D(p_0 \mid p^*) = \inf_{p \in P^*} D(p_0 \mid p),
\]

where $D(p_1 \mid p_2) = \sum_j p_j^1 \log \frac{p_j^1}{p_j^2}$ is KL divergence. If $p_j^* = 0$ for some $j$ in a regular game, $\sup \mathcal{K}_1 = \infty$.

• For $N$-step game, suppose that Reality adopts $\tilde{p}^0$. Define $\alpha_{in}^* = \alpha_i^{(j_1 \ldots j_{n-1})}$ by
\[
\sum_{j=1}^m \frac{a_{ij}}{1 + \sum_{h=1}^k \alpha_h^* a_{hj}} p_{j(j_1 \ldots j_{n-1})}^0 \leq 0, \quad \forall i
\]
Theorem 3

\[ E_{\tilde{\mathbf{p}}^0}(\log \mathcal{K}_N) = D(\tilde{\mathbf{p}}^0 | \tilde{\mathbf{p}}^*) = \inf_{\tilde{\mathbf{p}} \in \tilde{P}^*} D(\tilde{\mathbf{p}}^0 | \tilde{\mathbf{p}}). \]

When \( \tilde{\mathbf{p}}_0 = (\mathbf{p}_0)^N \) (\( N \)-fold direct product),

\[ E_{\tilde{\mathbf{p}}^0}(\log \mathcal{K}_N) = N D(\mathbf{p}_0 | \mathbf{p}^*) = N \inf_{\mathbf{p} \in P^*} D(\mathbf{p}_0 | \mathbf{p}). \]
The case of unknown $p_0$ (still in usual statistical setting)

- Let $\hat{p}_{nj}$ be an estimate of $p_j^0$ at round $n$.
- Define $\hat{\alpha}_ni$ and $\hat{p}_{nj}^*$ by

$$
(1 + \sum_h \hat{\alpha}_{nh} a_{hj}) \hat{p}_{nj}^* = \hat{p}_{nj}.
$$

- Then

$$
\log \mathcal{K}_N = \sum_{n=1}^N \log \frac{\hat{p}_{njn}}{\hat{p}_{nj}^*}
= \sum_{n=1}^N \log \frac{\hat{p}_{njn}}{p_j^*} - \sum_{n=1}^N \log \frac{\hat{p}_{njn}}{p_j^*}.
$$
• Bayes estimator

\[ \hat{p}_{nj} = \frac{m_{nj} + c}{n + mc}, \quad m_{nj} = \#\{j_l = j \mid 1 \leq l \leq n-1\}, c > 0. \]

•

\[
\sum_{n=1}^{N} \log \hat{p}_{nj} = \sum_{j=1}^{m} \log \Gamma(m_{nj} + c) - \log \Gamma(N + mc)
\]
\[ - m \log \Gamma(c) + \log \Gamma(mc). \]

• By Stirling’s formula, we can prove the following game-theoretic theorem.
Theorem 4

\[
\log K_N = N \sum_{j=1}^{m} \hat{p}_{Nj} \log \frac{\hat{p}_{Nj}}{\hat{p}_j} - \frac{m}{2} \log N - \sum_{n=1}^{N-1} \log \frac{\hat{p}_{n|n}^*}{p_{jn}^*} + O(1)
\]

\[
= ND(\hat{p} | p^*) - O(\log N)
\]

and Skeptic can force Reality to make \((\epsilon > 0 \text{ arbitrary})\)

\[
\limsup_N N^{-\frac{1}{2} - \epsilon} \sum_{n=1}^{N} a_{ijn} \leq 0, \quad \forall i.
\]

[Partly discussed in papers No.3 and No.10]
Let \( \{b_j\} \) an additional item in the unit game with price \( \pi \).

Write \( a_{k+1,j} = \frac{b_j}{\pi} - 1 \).

The best strategy \( \{\alpha_i^*\}, i = 1, \ldots, k + 1 \), for Skeptic satisfies

\[
\sum_j a_{ij} \frac{p_j^0}{1 + \sum_{h=1}^{k+1} \alpha_h^* a_{hj}} \leq 0, \quad i = 1, \ldots, k + 1.
\]

Putting \( (1 + \sum_{h=1}^{k+1} a_{hj} \alpha_h^*) p_j^* = p_j^0 \), we have

\[
\sum_j a_{ij} p_j^* \leq 0, \quad i = 1, \ldots, k \quad \text{and} \quad \sum_j b_j p_j^* \leq \pi.
\]
Define $p^*$ by $D(p^0 | p^*) = \inf_{p \in P^*} D(p^0 | p^*)$. Then

\[ \sum_j b_j p^*_j < \pi \Rightarrow \alpha^*_{k+1} = 0, \]
\[ \sum_j b_j p^*_j > \pi \Rightarrow \alpha^*_{k+1} > 0. \]
3. Sequential hypothesis testing
Sequential hypothesis testing

- The null hypothesis:
  \[ H : p \in P^*. \]

- Under \( H \), \( \mathcal{K}_n \) is a positive martingale.

- Let \( \Sigma \) be a stopping rule and let \( N \) be the associated stopping time.

- \( \Sigma(j_1, \ldots, j_n) = 0 \) or 1. As long as \( \Sigma = 0 \) Skeptic continues gambling and once \( \Sigma(j_1, \ldots, j_N) = 1 \) he stops gambling and retrieves \( \mathcal{K}_N \).

By the optional stopping theorem we have
Theorem 5  Under the null hypothesis

\[ E(K_N) = K_0, \quad P^*(K_N/K_0 \geq c) \leq \frac{1}{c}, \quad c > 0. \]

If we reject \( H \) when \( K_N/K_0 \geq c \), then the level of significance \( \alpha \) satisfies \( \alpha \leq 1/c. \)

- With Kelly’s strategy against \( p_0 \)

\[
\frac{K_N}{K_0} = \prod_{n=1}^{N} \frac{p_{n}^{0}}{p_{n}^{*}} = LR.
\]
Wald’s SPR test

- Let $\Sigma = 0$ for $d < \mathcal{K}_n < c$, reject $H$ if $\mathcal{K}_N \geq c$, and accept $H$ if $\mathcal{K}_N \leq d$.

- This test is identical with the SPR test and has the optimality of SPR test.
• For any capital process $\mathcal{K}_n$ with $\mathcal{K}_0 = 1$, a sequential test is obtained by:
  - Stop as soon as $\mathcal{K}_n \geq c$ or $\mathcal{K}_n \leq d$ happens.
  - In the former case reject $H$ and in the latter case accept $H$.
  - The approximate level of significance $\alpha$ is
    \[ \alpha \sim \frac{1 - d}{c - d}. \]

Example: multinomial distribution.

• $X_1, X_2, \ldots$, i.i.d. $P(X_n = j) = p_j$, $j = 1, \ldots, k$.

• $H : p_j = p_j^*$. 
• Corresponding game: \( A = \{a_{ij}\}, 1 \leq i, j \leq k. \)

\[
a_{ii} = \frac{1}{p_i^*} - 1, \quad a_{ij} = -1, \quad i \neq j.
\]

• Consider a test based on Bayes Kelly’s strategy

\[
\hat{\alpha}_{n,i} = \frac{m_{ni} + c}{n + mc}
\]

• With \( c = 1 \), the test statistic is given as

\[
\mathcal{K}_n = \frac{n!}{(n + k)!} \left/ \left( \frac{n!}{\prod_j m_{nj}!} \prod_j (p_j^*)^{m_{nj}} \right) \right.
\]
4. Asset trading games in continuous time

4.1 Formulation

4.2 $\eta$-step strategy [paper No.10, 11]

4.3 Strategies in induced $\eta$-step games
4.1 Formulation

- \( S(t), \, 0 \leq t \leq T \): the price of a tradable asset in continuous time.

- \( \text{T.V.}(S) \): total variation of \( S \) on \([0, T]\).

\[
\text{T.V.}(S) = \sup_{0=t_0<t_1<\ldots<t_N<t_{N+1}=T} \sum_{i=0}^{N} |S(t_{i+1}) - S(t_i)|
\]

(this could be infinite)
Protocol:

• Reality chooses a continuous positive function $S$ with $\text{T.V.}(S) \geq A$. ($A > 0$: given)

• Skeptic chooses trading times and number of assets to hold.
  - He chooses $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = T$, where $t_n$ can depend on $\{S(t) \mid t \leq t_n\}$.
  - At time $t_{n-1}$ he also chooses the number of units $M_n$ of the asset to hold, depending on $\{S(t) \mid t \leq t_{n-1}\}$.

• Then the capital $\mathcal{K}_n$ at time $t_n$ is written as

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n(S(t_n) - S(t_{n-1}))/S(t_{n-1}).$$
4.2 \( \eta \)-step strategy

- Choose \( t_0 = 0 < t_1 < \cdots < t_N < t_{N+1} = T \) by

\[
|\log S(t_{i+1}) - \log S(t_i)| = \eta, \quad i = 0, 1, \ldots
\]

\[
|\log S(t_{i+1}) - \log S(t_i)| < \eta \quad \text{for} \quad t_i < t < t_{i+1}
\]

\[
|\log S(t) - \log S(t_N)| < \eta \quad \text{for} \quad t_N < t < T.
\]

- Induced \( \eta \)-step game

\textbf{Protocol:} \( K_0 = 1. \)

\[
K_n = K_{n-1}(1 + \alpha_n x_n), \quad n = 1, \ldots, N,
\]

where \( x_n = e^n - 1 \) or \( e^{-n} - 1. \)
The optimal strategy for Skeptic is
\[ \alpha^* = \frac{e^n - 1}{e^n + 1} \] and for Reality
\[ p^* = \frac{1}{e^n + 1}. \]

**Theorem 6**  Optimal strategy for Reality for all induced \( \eta \)-step game is:

\[ S(t) \text{ is a positive martingale.} \]

This implies that there exists a path-dependent and future-independent time change \( t = t(\tau) \) such that
\[ \log S^*(\tau) = \log S(t(\tau)) \text{ is a Brownian motion with a drift.} \]
Upper price of a derivative $\phi(S(T))$ depending on $S(T)$.

- In the $\eta$-step game
  - For fixed $N$ \quad ($q^* = 1 - p^*$)
    \[ E_{p^*}(\phi(S(T))) \sim \int \frac{1}{\sqrt{2\pi N p^* q^*}} \phi(e^u) \exp\left(-\frac{u^2}{2N p^* q^*}\right) du \]
  - Since $N$ depends on the path,
    \[ E_{p^*}(\phi(S(T))) \sim \sup E_N \left( \int \frac{1}{\sqrt{2\pi N p^* q^*}} \phi(e^u) \exp\left(-\frac{u^2}{2N p^* q^*}\right) du \right), \]
    \text{where $\sup E_N$ means supremum for all possible distribution of $N$.}
Upper price of \( \phi \) in the original game:

\[
\sup E\sigma^2 \left( \int \frac{1}{\sqrt{2\pi}\sigma^2} \phi(e^u) \exp\left(-\frac{u^2}{2\sigma^2}\right) du \right),
\]

where supremum is calculated for all possible distribution of \( \text{Var}(\log(S(T)/S(0))) \).
4.3 Strategies in induced $\eta$-step games

1) One stage strategy

- If Skeptic assumes that Reality is stochastic and her distribution is i.i.d. $p = \Pr(x_n = e^n - 1)$, then Skeptic’s best strategy is

$$\alpha^* = \frac{pe^n - q}{e^n - 1} \quad (q = 1 - p).$$
• Under this Skeptic’s strategy the capital process is

$$\log K_n = m_n \log \frac{p}{p^*} + (n - m_n) \log \frac{q}{q^*},$$

where $m_n = \#\{h \mid x_h = e^n - 1, 1 \leq h \leq n\}$ is the number of heads. This formula is path-wise true.

• When Skeptic does not want to specify $p$, he can use the Bayesian strategy

$$\hat{\alpha}_n^* = \frac{\hat{p}_{n,c} e^n - \hat{q}_{n,c}}{e^n - 1}, \quad \hat{p}_{n,c} = \frac{m_n + c}{n + 2c}, \quad c > 0.$$
Proposition 9  Let $\hat{p}_N = m_N/N$. Under the Bayesian strategy, Skeptic’s capital is written as

$$\log K_N = N D(\hat{p}_N \mid p^*) - \frac{1}{2} \log N + O(1).$$

If $\hat{p}_N$ is close to $p^*$, then $\log K_N$ is further approximated as

$$\log K_N = \frac{N}{2} \frac{(\hat{p}_N - p^*)^2}{\hat{p}_N \hat{q}_N} - \frac{1}{2} \log N + O(1).$$

- $N$ depends on the path $S(t), 0 \leq t \leq T$, and $\eta$.
- We consider letting $\eta \downarrow 0$. Write $N_\eta$. 
\[
\eta(2m_{N_\eta} - N_\eta) = \log S(T) - \log S(0) = L, \\
\log K_N \sim \frac{N_\eta}{8} (\eta + \frac{L}{\eta N_\eta})^2 - \frac{1}{2} \log N_\eta + O(1) \\
= \frac{\eta^2 N_\eta}{8} + \frac{L^2}{8 \eta^2 N_\eta} - \frac{1}{2} \log N_\eta + O(1).
\]

- It follow that if \( N_\eta = O(\eta^{-2-\epsilon}) \) or if \( L \neq 0 \) and \( N_\eta = O(\eta^{-2+\epsilon}) \) for some \( \epsilon > 0 \), then \( K_{N_\eta} \to \infty \) as \( \eta \to 0 \).

- This implies \( \sqrt{dt} \) effect, i.e., the variation exponent of \( S(t) \) is 2.
1) Two stage Markov strategy

- Consider the sign pattern of Realities moves $x_n$, e.g.
  $+ - - + - - + - \cdots$

- Let $\hat{\alpha}_n^*$ depend on the sign of $x_{n-1}$.

  $$\hat{\alpha}_n^* = \begin{cases} 
  \hat{\alpha}_{n+}^* & \text{if } x_{n-1} > 0, \\
  \hat{\alpha}_{n-}^* & \text{if } x_{n-1} < 0, 
  \end{cases}$$

  where

  $$\hat{\alpha}_{n\pm}^* = \frac{\hat{p}_{n,c}^\eta - \hat{q}_{n,c}^\pm}{\epsilon^\eta - 1}.$$
• Let

\[ n_1 = \#(++) \] \[ n_2 = \#(+-) \] \[ n_3 = \#(-+) \] \[ n_4 = \#(--) \].

• Then by counting hitting times of grids with the grid size of \( \eta \) within the grids of size \( 2\eta \), we have the following relations.

\[ n_1 + n_4 = N_{2\eta} \pm 1, \quad 2(n_1 + n_2 + n_3 + n_4) = N_\eta, \]
\[ 2\eta(n_1 - n_2) = L, \quad n_2 = n_3 \pm 1, \]
\[ n_1 = \frac{1}{2}(N_{2\eta} + \frac{L}{2\eta}), \quad n_4 = \frac{1}{2}(N_{2\eta} - \frac{L}{2\eta}), \]
\[ n_2 = n_3 = \frac{1}{2}(\frac{N_\eta}{2} - N_{2\eta}). \]
Proposition 10  Under the two stage Markov strategy the capital process is written as

\[
\log \mathcal{K}_{N_t} = (n_1 + n_2)D(\hat{p}_+ | p^*) + (n_3 + n_4)D(\hat{p}_- | p^*) \\
- \frac{1}{2} \log(n_1 + n_2) - \frac{1}{2} \log(n_3 + n_4) + O(1),
\]

where

\[
\hat{p}_+ = \frac{n_1}{n_1 + n_2}, \quad \hat{p}_- = \frac{n_3}{n_3 + n_4}.
\]
• As $\eta \downarrow 0$,

$$\log \mathcal{K}_{N_\eta} = \frac{1}{8}(N_\eta + \frac{L}{\eta})(\hat{p}_+ - p^*)^2 + \frac{1}{8}(N_\eta - \frac{L}{\eta})(\hat{p}_- - p^*)^2$$

$$- \frac{1}{2} \log(N_\eta^2 - \frac{L^2}{\eta^2}) + O(1).$$

• $\log \mathcal{K}_{N_\eta} \to \infty$ as $\eta \to 0$ if for some $\epsilon > 0$

  either $N_\eta \eta^{2+\epsilon} \to \infty$ or $N_\eta \eta^{2-\epsilon} \to 0$.

• With Markov strategy, we can also describe how fast $\mathcal{K}_{N_\eta}$ diverges when the variation exponent of $S(t)$ deviates from 2.